Intrinsic Dynamics of Manifolds: Quantum Paths, Holonomy, and Trajectory Localization

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Abstract

We consider a dynamic generalization of the classical "kinematic" notion of affine connection providing a correspondence between paths in the manifold and diffeomorphisms of the manifold. The linearization of these path-diffeomorphisms coincides with the parallel translations via the connection. In this dynamic geometry one can translate nonanalytic functions and distributions rather than tangent vectors. We describe the dynamic holonomy and the dynamic curvature.

On the symplectic or quantum level this construction makes up the symplectic or quantum paths, as well quantum connection, quantum curvature and quantum holonomy.

The construction of path-diffeomorphisms, being applied to trajectories of a given dynamical system, produces a dynamic localization of the system. In the symplectic (quantum) case, this dynamic localization provides a coherent-type representation of the quantum flow.

1 Introduction

Kinematics of manifolds is a parallel translation of tangent vectors along paths. The notion of affine connection provides the mathematical descrip-

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tion of this mechanical concept. Usual constructions of the connection theory (exponential mappings, geodesics, curvature, holonomy, etc.) are objects of the infinitesimal geometry. There are no potential forces in this geometry and thus no opportunities to translate nonanalytic functions or distributions along paths. Each path in the manifold knows only about the germ of the geometry but not about the neighborhood geometry of the manifold. There is no place for the intrinsic wave concept in this kinematic approach. That is why one would like to "integrate" the usual connection ideology to a more substantial framework, which can be called a *dynamic geometry*. In particular, this generalization is necessary for solving the quantization problem [1].

Dynamics is a translation of functions or distributions. The intrinsic dynamics of a manifold, which we have in mind, is generated by certain internal non-autonomous vector field whose "time" variable ranges over the same manifold. In the symplectic case this internal vector field is Hamiltonian; the corresponding Hamilton function determines the Ether structure [1].

In general, the internal field is a family of linear maps acting in pairs of spaces tangent to the manifold. This family can also be considered as a one-form with values in vector fields, that is, in the Lie algebra of first-order differential operators on the manifold. The basic condition for this form is the equation of zero curvature.

After restriction to the diagonal, where the "time" variable coincides with the "space" variable, this zero curvature equation occurs to be the Cartan structural equation (and its solvability is guaranteed by the 1-st Bianchi's identity). Thus the idea of the internal field is a natural extension of Cartan's "moving frame" approach. The main goal and advantage of such an extension is to get all charms and conveniences of zero curvature over manifolds with a nonflat affine connection.

Because of the zero curvature condition, the internal translation of functions (distributions, curves, surfaces, etc.) along any path in the manifold does not depend on the shape of the path. In this way we obtain a big family of diffeomorphisms generating the first level of intrinsic dynamics of the manifold.

In the symplectic case, these diffeomorphisms are symplectic, and among them there are Ether translations and reflections [1]. They are applied to calculation of quantum geometric objects and determine, for instance, the membrane representation of functions and the geometric phase product [2].

In the given paper we consider a second level of intrinsic dynamics. It is generated by the internal vector field with the factor $\frac{1}{2}$. In presence of this

factor one looses the zero curvature condition: the translation along a path now depends on the shape of the path and determines a transformation of the manifold. We obtain a realization of the path groupoid by diffeomorphisms.

Each path-diffeomorphism maps the origin of the path to its end. The linearization of the path-diffeomorphism along the path itself coincides with the parallel translation by means of a connection in the tangent bundle. This is a point. We see that the path-diffeomorphism can be considered as a dynamic generalization of the usual kinematic parallel translation of vectors. In the symplectic framework, this generalization provides a notion of symplectic path and quantum path.

Applying the construction of path-diffeomorphisms and considering closed paths, at each given point we obtain a *dynamic holonomy* realized by diffeomorphisms with the given fixed point. Let us stress that this is not the habitual holonomy in the tangent bundle. Only the linearization at the fixed point gives us the usual kinematic holonomy.

The dynamic generalization of the notion of holonomy opens the opportunity to obtain (in the symplectic case) the construction of quantum holonomy, say, following the quantization approach [1, 3], which allows one to quantize not only functions, but also symplectic transformations.

In the standard way one can calculate the Lie algebra of the dynamic holonomy group. This Lie algebra is related to certain *dynamic curvature*. The germ of the dynamic curvature at the fixed point is given by the ordinary kinematic curvature (and by the symplectic form if we are inside the symplectic framework).

At the next stage we consider a certain external dynamical (Hamiltonian) system on the manifold. The trajectories of this system are paths, and to them one can assign path-diffeomorphisms (symplectomorphisms) of the manifold. Each diffeomorphism can be applied to any function or to any dynamical system on the manifold. In particular, if it is applied to the given system itself, then one obtains a new system that has the equilibrium point at the origin of the trajectory.

In a sense, this is a procedure of localization of the dynamical system at the given trajectory. That is why we use the term "localization" in the title of this paper. But we would like to stress that the new transformed system not only inherits the germ (say, the first variation) of the original system at the chosen trajectory, but provides the total global information about the original system. For this type of transformations, we use the name *translocations*.

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The translocation is determined by a chosen trajectory of a dynamical system and by the intrinsic dynamics of the whole manifold. This is a representation of the system via the intrinsic travelling stream generated by the internal vector field on the manifold.

In the symplectic case the translocation reduces a given Hamiltonian system to a new one with a Hamilton function approximately quadratic near the given point. The corresponding quantum transformation is well known in the Euclidean space and is extensively used in the oscillatory approximation for wave and quantum equations(see, for instance, [4]). In the case of general symplectic manifolds, the existence of such transformations was in question. The concept of Ether structure [1] allows one to answer this question.

In this paper we consider both the classical and quantum aspects of these problems and develop the *dynamic geometry* framework following the ideology of [1]. The symplectic case is of our main interest here, and therefore the basic text of the paper deals with this case. The Appendix contains a parallel description of the general situation (manifolds with affine connection).

2 Ether dynamics

Let $\mathfrak{X}$ be a manifold with a symplectic form ω, and with a torsion free symplectic connection Γ. By $\{f, g\} = \nabla f \cdot \Psi \cdot \nabla g$ we denote the Poisson brackets related to ω, and by ∇ the covariant derivative with respect to Γ.

The *Ether structure* on $\mathfrak{X}$ is given by a certain intrinsic Hamiltonian $\mathcal{H}$ which is a 1-form on $\mathfrak{X}$ with values in smooth real functions on $\mathfrak{X}$. In local coordinates:

$$\mathcal{H}_x(z) = \sum_{j=1}^{2n} \mathcal{H}_x(z)_j \, dx^j, \qquad 2n = \dim \mathfrak{X}, \quad x, z \in \mathfrak{X}.$$

The following conditions should hold:

(a) the zero curvature condition

(2.1)
$$\partial \mathcal{H} + \frac{1}{2} \{\mathcal{H} \underset{,}{\wedge} \mathcal{H}\} = 0$$

(where $\partial = d_x$ is the differential by x and the Poisson brackets are taken by z);

(b) the boundary conditions

(2.2) $$\mathcal{H}\Big|_{\text{diag}} = 0, \qquad \nabla\mathcal{H}\Big|_{\text{diag}} = 2\omega, \qquad \nabla\nabla\mathcal{H}\Big|_{\text{diag}} = 0$$

(where $\nabla = \nabla_z$ and diag $= \{z = x\}$);
(c) the skew-symmetry condition

(2.3) $$\mathcal{H}_x\big(s_x(z)\big) = -\mathcal{H}_x(z),$$

where $s = s_x(z)$ is the solution of the Ether Hamiltonian system

(2.4) $$\partial s = \nabla\mathcal{H}(s)\Psi(s), \qquad s\Big|_{x=z} = z.$$

In the last system $\partial = d_x$ and the variable $x \in \mathfrak{X}$ plays the role of time.

Theorem 2.1. (i) *For any symplectic connection on a symplectic manifold $\mathfrak{X}$, the zero curvature equation (2.1) with conditions (2.2), (2.3) is solvable at least in a neighborhood of the diagonal* diag $= \{z = x\}$.
(ii) *The mappings s_x defined by (2.4) introduce a reflective structure to $\mathfrak{X}$:*
$-$ *s_x are symplectic,*
$-$ *$s_x^2 = $ id,*
$-$ *s_x has the isolated fixed point x.*
(iii) *By a given reflective structure on the simply connected symplectic manifold $\mathfrak{X}$, one reconstructs the corresponding symplectic connection and the Ether Hamiltonian as follows:*

$$\Gamma_{kl}^j(x) = -\frac{1}{2}\frac{\partial^2 s_x^j(z)}{\partial z^k \partial z^l}\bigg|_{z=x},$$

$$\mathcal{H}_x(z)_k = \int_x^z \frac{\partial s_x^j}{\partial x^k}\big(s_x(z)\big)\omega_{jm}(z)\, dz^m,$$

where the integral is taken along any path connecting x with z.
(iv) *In addition to the boundary conditions (2.2), the third covariant derivative of $\mathcal{H}_x(z)$ by z on the diagonal $z = x$ is related to the curvature tensor R of the connection Γ:*

$$\nabla_j\nabla_l\nabla_s\mathcal{H}_k\Big|_{\text{diag}} = 2\omega_{sm}R_{lkj}^m.$$

All higher covariant derivatives of $\mathcal{H}$ on the diagonal are also explicitly evaluated in terms of ω and R.

Note that the zero curvature condition (2.1) guarantees the solvability of (2.4). Besides of (2.4), one can consider more general Hamiltonian system

$$(2.5) \qquad \frac{d}{dt}G = \dot{y}(t)\nabla \mathcal{H}_{y(t)}(G)\Psi(G), \qquad G\Big|_{t=0} = x.$$

Here $\{y(t) \mid 0 \leq t \leq 1\}$ is a smooth path connecting a point $y_0 = y(t)|_{t=0}$ with a point $y_1 = y(t)|_{t=1}$. By $\dot{y}(t)$ we denote the velocity vector at the point $y(t)$ of the path.

The zero curvature condition (2.1) implies that the solution $G = G(t)$ of (2.5) at $t = 1$ does not depend on the shape of the path $\{y(t)\}$, that is, we can denote:

$$G\Big|_{t=1} = g_{y_1,y_0}(x).$$

Thus one obtains a family of symplectic transformations g_{y_1,y_0} of the manifold $\mathfrak{X}$. We call them [1] the *Ether translations*. They are related to the reflections as follows.

Theorem 2.2. $\quad g_{y,x} = s_y s_x, \quad g_{y,x}(x) = s_y(x).$

3 Factor $\frac{1}{2}$

The symplectic connection Γ has its geodesics and exponential mappings $\exp_x$, defined in the standard way. The Ether Hamiltonian $\mathcal{H}$, which provides an "integral" approach to the connection theory on $\mathfrak{X}$, generates its own geodesics and exponential mappings by means of the Hamiltonian translations. Namely, let us consider the Hamiltonian system [1]:

$$(3.1) \qquad \frac{d}{dt}E = \frac{1}{2}v\nabla \mathcal{H}_x(E)\Psi(E), \qquad E\Big|_{t=0} = x.$$

Here $v \in T_x\mathfrak{X}$ is a fixed velocity vector. Denote the solution of (3.1) by

$$E(t) = \mathrm{Exp}_x(vt).$$

This trajectory we call the *Ether geodesics* through x, and Exp_x is called the Ether exponential mapping. This mapping, in general, is different from $\exp_x$, and the Ether geodesics do not coincide with the Γ-geodesics.

The following relations hold:

$$s_x\big(\operatorname{Exp}_x(v)\big) = \operatorname{Exp}_x(-v),$$
$$\mathcal{H}_x\big(\operatorname{Exp}_x(v)\big) = -\mathcal{H}_x\big(\operatorname{Exp}_x(-v)\big).$$

Let us now pay attention to the factor $\frac{1}{2}$ on the right of (3.1). We can consider more generic systems with such a factor:

$$(3.2) \qquad \frac{d}{dt}Y = \frac{1}{2}\dot{y}(t)\nabla\mathcal{H}_{y(t)}(Y)\Psi(Y), \qquad Y\Big|_{t=0} = x.$$

Here $\{y(t)\}$ is a given smooth path in $\mathcal{H}$. The solution of (3.2) we denote by $Y = Y^t(x)$; thus Y^t is the translation along the trajectories of (3.2).

Of course, the mapping Y^t, determined by the Hamiltonian system, is symplectic. But in the presence of the factor $\frac{1}{2}$ in (3.2), we loose the zero curvature property, and so, in contrast to (2.5), the mapping Y^t does depend on the shape of the path $\{y(t)\}$. Let us formulate the key observation about this mapping.

Theorem 3.1. *The following identities hold*:

$$Y^t(y(0)) = y(t), \qquad dY^t(y(0)) = V^t.$$

Here V^t is the parallel translation $T_{y(0)}\mathfrak{X} \to T_{y(t)}\mathfrak{X}$ by means of the connection Γ along the path $\{y(\tau) \mid 0 \leq \tau \leq t\}$.

Proof. In view of the boundary conditions (2.2), we have

$$\frac{1}{2}\nabla\mathcal{H}_y(Y)\Psi(Y) = I + \langle\Lambda, \delta Y\rangle + O(\delta Y^2).$$

Here $\delta Y = Y - y$ in some local coordinates, and $\Lambda = \omega(\partial\Psi + \Gamma\Psi)$. Taking the symplecticity of the connection Γ into account, the system (3.2) can be written as

$$(3.3) \qquad \frac{d}{dt}\delta Y + \dot{y}\Gamma(y)\,\delta Y + O(\delta Y^2) = 0.$$

If the initial data for Y coincides with the initial data for y, then $\delta Y|_{t=0} = 0$, and from (3.3) we have $\delta Y = 0$ for any t. Thus the first identity of the theorem is true.

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From (3.3) it also follows that the differential $V^t = dY^t(y(0))$ obeys the equation

$$\frac{d}{dt}V^t + \dot{y}(t)\Gamma(y(t))\,V^t = 0.$$

This means that V^t is the parallel translation along the path $\{y(\tau) \mid 0 \leq \tau \leq t\}$ by means of the connection Γ. The theorem is proved.

This theorem explains why we take the factor $\frac{1}{2}$ on the right-hand side of (3.2). Any different choice destroys the statement of the theorem.

4 Symplectic paths

Let us introduce more convenient notation. A path in $\mathfrak{X}$ we will denote by $\sigma = \{y(t)\}$, and the translation along trajectories of (3.2) denote by $[\sigma] = Y^t$. So, $[\sigma](y(0)) = Y$ is the solution of (3.2).

Definition 4.1. The symplectic transformation $[\sigma] : \mathfrak{X} \to \mathfrak{X}$ we call the *symplectic path* corresponding to the classical path $\sigma \subset \mathfrak{X}$.

Consider the induced mapping $[\sigma]^* : C^\infty(\mathfrak{X}) \to C^\infty(\mathfrak{X})$. From equation (3.2) we have the following explicit formula:

$$(4.1) \qquad\qquad [\sigma]^* = \mathop{\mathrm{Exp}}_{\rightarrow}\left(\frac{1}{2}\int_\sigma \mathrm{ad}(\mathcal{H})\right).$$

Here $\mathrm{ad}(f)$ denotes the Hamiltonian vector field on $\mathfrak{X}$ related to a function f, and $\mathop{\mathrm{Exp}}_{\rightarrow}$ is the multiplicative exponential ordered from left to right:

$$(4.2) \qquad\qquad \mathop{\mathrm{Exp}}_{\rightarrow}\left(\int_0^t a\right) = \lim_{N\to\infty} e^{\Delta t\, a(t_1)} \dots e^{\Delta t\, a(t_N)},$$

where $\Delta t = t/N$ and $t_j = j\Delta t$. Formula (4.1) represents the parallel translation along the path σ with respect to the connection

$$(4.3) \qquad\qquad \nabla^0 = \partial + \frac{1}{2}\,\mathrm{ad}(\mathcal{H})$$

acting in the trivial bundle over $\mathfrak{X}$ with fibers $C^\infty(\mathfrak{X})$.

Theorem 4.2. (i) *The correspondence $\sigma \rightarrow [\sigma]$ between classical paths and symplectic paths is given by formula* (4.1). *This is a realization of the path groupoid in the group of symplectic transformations of $\mathfrak{X}$: if the product of two paths $\sigma_2 \circ \sigma_1$ exists, then*

$$[\sigma_2 \circ \sigma_1] = [\sigma_2] \circ [\sigma_1].$$

(ii) *The differential $d[\sigma]$ of the symplectic path at the origin of the path coincides with the parallel translation along the path σ by means of the connection Γ.*

(iii) *Symplectic paths commute with reflections, i.e., if $\sigma_{y,x}$ is a path connecting x with y, then*

(4.4) $$[\sigma_{y,x}] \circ s_x = s_y \circ [\sigma_{y,x}].$$

In particular, the mapping $[\sigma_{y,x}]$ is affine along the path $\sigma_{y,x}$, that is, it translates the connection coefficients $\Gamma(x)$ to $\Gamma(y)$.

5   Symplectic loops and Ether curvature

Now, for any point $x \in \mathfrak{X}$, we can consider the group of closed paths σ (parametrized loops) starting from x. The corresponding symplectic loops $[\sigma]$ form a subgroup in the whole group of symplectic transformations of the manifold $\mathfrak{X}$ having x as a fixed point. This subgroup we call a *dynamic holonomy group* and denote by $\mathcal{J}_x$.

The differentials $d[\sigma](x)$ of symplectic loops at the fixed point x are symplectic linear transformations of $T_x\mathfrak{X}$. They form a subgroup in the whole group of linear transformations of $T_x\mathfrak{X}$. This subgroup coincides with the usual holonomy group of the connection Γ at the point x. The latter group we shall call *kinematic* in order to distinguish it from the dynamic holonomy group.

Obviously, the dynamic holonomy groups $\mathcal{J}_x$ and $\mathcal{J}_y$ corresponding to different points x and y are gauge equivalent (conjugate to each other):

$$\mathcal{J}_x = [\sigma_{y,x}]^{-1} \circ \mathcal{J}_y \circ [\sigma_{y,x}],$$

where $\sigma_{y,x}$ is a path connecting x with y.

Let us mark a certain point $0 \in \mathfrak{X}$. Consider small membranes $\Sigma \subset \mathfrak{X}$ whose boundaries are parametrized loops $\partial\Sigma \in \mathcal{J}_0$.

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Note that the multiplicative exponential (4.2) can be represented (using the continual version of the Campbell–Hausdorff formula) as follows:

$$\underset{\rightarrow}{\mathrm{Exp}}\left(\int_{\partial\Sigma} a\right) = \exp\left(\int_{\Sigma} b + o(\Sigma)\right).$$

Here $b = da + \frac{1}{2}[a \wedge a]$ is the curvature of a, and the summands $o(\Sigma)$ are of higher degree with respect to the area of Σ.

Applying this fact to (4.1), we see that

$$[\partial\Sigma]^{-1*} = \exp\left(-\frac{1}{2}\int_{\Sigma} \mathrm{ad}\left(\partial\mathcal{H} + \frac{1}{4}\{\mathcal{H} \underset{,}{\wedge} \mathcal{H}\}\right) + o(\Sigma)\right).$$

The sign minus in the exponent is due to the additional inversion of the map $[\partial\Sigma]$; we need it to have the group homomorphism $\partial\Sigma \to [\partial\Sigma]^{-1*}$.

Taking into account the zero curvature condition (2.1), we derive

(5.1) $$[\partial\Sigma]^{-1*} = \exp\left(\frac{1}{8}\int_{\Sigma} \mathrm{ad}(\{\mathcal{H} \underset{,}{\wedge} \mathcal{H}\}) + o(\Sigma)\right).$$

These are operators in $C^{\infty}(\mathfrak{X})$ representing the dynamic holonomy group $\mathcal{J}_0$.

Near the unity element $0 \in \mathcal{J}_0$ any loop $\sigma \in \mathcal{J}_0$ can be considered as a plane one. The differential of the family of operators (5.1) at the unity point of the group $\mathcal{J}_0$ is the operator-valued 2-form

(5.2) $$d[\sigma]^{-1*}\Big|_{\sigma=0} = \frac{1}{8}\,\mathrm{ad}(\{\mathcal{H}_0 \underset{,}{\wedge} \mathcal{H}_0\}).$$

Here $\mathcal{H}_0$ is the Ether Hamiltonian at the marked point $0 \in \mathfrak{X}$.

Let us fix certain local coordinates near the marked point $0 \in \mathfrak{X}$. For any small $\sigma \in \mathcal{J}_0$ denote by σ_{jk} the area enclosed by the projection of σ onto (j, k)-coordinate plane (with the orientation given by $dx^k \wedge dx^j$). Denote by $\mathcal{H}_{0j}$ the components of the Ether Hamiltonian $\mathcal{H}_0$ with respect to the chosen local coordinates, and introduce the following functions:

(5.3) $$\mathcal{R}_{jk} \overset{\text{def}}{=} \frac{1}{4}\{\mathcal{H}_{0k}, \mathcal{H}_{0j}\}.$$

Note that from the zero curvature equation (2.1) and from definition (4.3) of the connection ∇^0 we know that the Hamiltonian fields of the functions $\mathcal{R}_{jk}$ represent the curvature of ∇^0, i.e.,

$$[\nabla^0_j, \nabla^0_k] = \mathrm{ad}(\mathcal{R}_{jk}).$$

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From (5.2) one has the formula

(5.4)
$$\left.\frac{\partial}{\partial \sigma_{jk}}[\sigma]^{-1*}\right|_{\sigma=0} = \mathrm{ad}(\mathcal{R}_{jk}).$$

Thus we obtain the following statement.

Theorem 5.1. *If $\mathfrak{X}$ is simply connected, then the Lie algebra of the dynamic holonomy group $\mathcal{J}_0$ at the marked point $0 \in \mathfrak{X}$ is generated by Hamiltonian vector fields related to functions $\mathcal{R}_{jk}$ that are defined in (5.3).*

The marked point is a stationary point of the Hamiltonians $\mathcal{R}_{jk}$, namely:

$$\mathcal{R}_{jk}(0) = \omega_{jk}(0), \qquad \nabla \mathcal{R}_{jk}(0) = 0.$$

Their second derivatives are given by the curvature tensor R of the connection Γ as follows:
$$\nabla^2_{lm} \mathcal{R}_{jk}(0) = 2\omega_{ls}(0) R^s_{mjk}(0).$$

We note that the curvature matrices $R_{jk}(0) = (\!(R^s_{mjk}(0))\!)$ represent operators in $T_0\mathfrak{X}$ which are skew-symmetric with respect to the symplectic inner product generated by $\omega(0)$, that is, they belong to the Lie algebra $\mathrm{sp}(T_0\mathfrak{X})$.

Corollary 5.2. *The Lie algebra of the symplectic holonomy group of $\mathfrak{X}$ at the marked point 0 is generated by components of the curvature form $R_{jk}(0) \in \mathrm{sp}(T_0\mathfrak{X})$.*

This statement is very common for the "kinematic" connection theory: it is a particular case of the general Ambrose–Singer theorem and is analogous to of E. Cartan's theorem about the Riemannian holonomy. In Theorem 5.1 we have its generalization to the dynamic holonomy group.

We shall call the form $\mathcal{R} = \frac{1}{8}\{\mathcal{H} \underset{,}{\wedge} \mathcal{H}\}$ staying in formula (5.1) the *Ether curvature form.*

6   Translocations

Let us consider now, on the symplectic manifold $\mathfrak{X}$, an exterior Hamiltonian system related to a smooth function H:

(6.1)
$$\frac{d}{dt}X = \nabla H(X)\,\Psi(X), \qquad X\bigg|_{t=0} = x.$$

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Denote the solution by $X = X^t(x)$; thus X^t is the Hamilton flow generated by H.

Let us fix a point $y \in \mathfrak{X}$ and a time $t \in \mathbb{R}$. We shall use the same notation both for the point $X^t(y)$ of the trajectory and for the whole segment of the trajectory:

(6.2)
$$X^t(y) \sim \{X^\tau(y) \mid 0 \leq \tau \leq t\}.$$

The symplectic path corresponding to the path (6.2) is denoted by $[X^t(y)]$ and will be called a *symplectic trajectory*. For each fixed t, it is a symplectic transformation of $\mathfrak{X}$.

We can define a new Hamilton function

(6.3)
$$H_y^t \overset{\text{def}}{=} [X^t(y)]^* \left(H - \frac{1}{2} \dot{X}^t(y) \, \mathcal{H}_{X^t(y)} \right) - H(y).$$

Related Hamiltonian system is

(6.4)
$$\frac{d}{dt} Z = \nabla H_y^t(Z) \, \Psi(Z), \qquad Z \Big|_{t=0} = x.$$

Denote the solution of this system by $Z = Z_y^t(x)$; thus Z_y^t is the translation along trajectories of (6.4).

Theorem 6.1. (i) *The translation X^t along trajectories of the original Hamiltonian system* (6.1) *and the translation Z_y^t along trajectories of the transformed Hamiltonian system* (6.4) *relate to each other as follows:*

(6.5)
$$X^t = [X^t(y)] \circ Z_y^t.$$

Here $y \in \mathfrak{X}$ is an arbitrary point, and $[X^t(y)]$ is the symplectic trajectory corresponding to the segment (path) (6.2).

(ii) *The point y is a stationary point of the Hamiltonian H_y^t of system* (6.4), *namely:* $H_y^t(y) = 0$, $(\nabla H_y^t)(y) = 0$. *The second derivative matrix at y is*

(6.6)
$$\nabla^2 H_y^t(z) \Big|_{z=y} = V_y^{t*} \cdot \nabla^2 H\big(X^t(y)\big) \cdot V_y^t.$$

Here $\nabla^2 H$ is the tensor of second covariant derivatives of H:

(6.7)
$$\nabla_{jk}^2 H \overset{\text{def}}{=} D_{jk}^2 H - D_m H \cdot \Gamma_{jk}^m,$$

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and V_y^t is the parallel translation along the path (6.2) *by means of the connection* Γ.

(iii) *The symplectic mapping* Z_y^t *in* (6.5) *has the fixed point* y. *The symplectic mapping* $[X^t(y)]$ *transports* y *to* $X^t(y)$.

The differential of the solution to system (6.1) *at the point* y *is given by*

$$(6.8) \qquad\qquad dX^t(y) = V_y^t \circ W_y^t.$$

Here $W = W_y^t$ *is the solution of the linear system over* $T_y\mathfrak{X}$:

$$(6.9) \qquad\qquad \frac{d}{dt}W = M_y^t W, \qquad W\Big|_{t=0} = I,$$

where

$$(6.10) \qquad\qquad M_y^t \stackrel{\text{def}}{=} -\Psi(y) \cdot V_y^{t*} \cdot \nabla^2 H(X^t(y)) \cdot V_y^t.$$

The transformation from the system (6.1) to the system (6.4) will be called a *translocation* to y. This is a dynamic analog of the usual kinematic parallel translation.

The translocation localizes the whole system near the given trajectory starting at y. In particular, the first variation of the original system along the trajectory is reduced to the linear equation (6.9) whose matrix is determined by the second covariant derivatives of H via (6.10) (about transformations of the first variation system via symplectic connections see a detailed investigation in [5]).

Corollary 6.2. *Let the Hamilton function* H *be covariantly quadratic along the trajectory* (6.2), *i.e.,*

$$(6.11) \qquad\qquad \nabla_{\nabla H\Psi}(\nabla^2 H) = 0.$$

at all points of the trajectory (6.2). *Then the solution of the first variation system along this trajectory is given by*

$$(6.12) \qquad\qquad dX^t(y) = V_y^t \circ \exp\big(-t\Psi(y)\nabla^2 H(y)\big).$$

If the trajectory (6.2) *is periodic, then its monodromy matrix at* y *is factorized to the product of the geometric monodromy* (*the holonomy of the connection*) *and the dynamical monodromy* (*with the generator* $-\Psi(y)\nabla^2 H(y)$).

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7 Quantum paths and quantum curvature

We assume that there is a representation of a space of functions on $\mathfrak{X}$ in a Hilbert space given by the integral

$$(7.1) \qquad f \to \hat{f} = \frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} f\mathbf{S}\, dm.$$

Here $\hbar > 0$, $2n = \dim\mathfrak{X}$, dm is a measure on $\mathfrak{X}$, and $\mathbf{S} = \{\mathbf{S}_x \mid x \in \mathfrak{X}\}$ is a family of operators in the Hilbert space obeying the Schrödinger type dynamics equation over $\mathfrak{X}$:

$$(7.2) \qquad i\hbar\partial\mathbf{S} = \widehat{\mathcal{H}}^{\hbar}\mathbf{S},$$

where $\mathcal{H}^{\hbar}$ is a quantum intrinsic Hamiltonian such that the quantum zero curvature equation holds:

$$(7.3) \qquad \partial\widehat{\mathcal{H}}^{\hbar} + \frac{i}{2\hbar}[\widehat{\mathcal{H}}^{\hbar} \underset{,}{\wedge} \widehat{\mathcal{H}}^{\hbar}] = 0.$$

As in the work [1], in addition to (7.2), (7.3), we assume that $\widehat{\mathcal{H}}^{\hbar}$ is self-adjoint and all $\mathbf{S}_x$ are self-adjoint and almost unitary operators:

$$(7.4) \qquad (\widehat{\mathcal{H}}^{\hbar})^* = \widehat{\mathcal{H}}^{\hbar}, \qquad \mathbf{S}_x^* = \mathbf{S}_x, \qquad \mathbf{S}_x^2 = \mu^2 \cdot I,$$

where $\mu = \mathrm{const}$. Note that the first and third conditions in (7.4) are not necessary, but we fix them to simplify all formulas. Under these conditions the term of order $\hbar$ in the expansion of the quantum Hamiltonina $\mathcal{H}^{\hbar}$ is absent; namely, we have

$$\mathcal{H}^{\hbar} = \mathcal{H} + O(\hbar^2),$$

where $\mathcal{H}$ is a classical intrinsic Hamiltonian from Sect. 2.

Now the first question arises: what is the quantum version of reflections s_x and Ether exponential mappings Exp_x were describe in [1], and the semiclassical asymptotic formulas were derived.

Now the first question arises: what is the quantum version of path-symplectomorphisms?

The answer is:

$$(7.5) \qquad \widehat{U}_{\sigma} = \underset{\leftarrow}{\mathrm{Exp}}\left(-\frac{i}{2\hbar}\int_{\sigma}\widehat{\mathcal{H}}^{\hbar}\right).$$

This unitary operator acting in the Hilbert space will be called the *quantum path* corresponding to the classical path $\sigma \subset \mathfrak{X}$. It is generated by the *quantum connection*

$$\boldsymbol{\nabla}^\hbar = \partial + \frac{i}{2\hbar}\widehat{\mathcal{H}}^\hbar.$$

Of course, the correspondence $\sigma \to \widehat{U}_\sigma$ is a representation of the path groupoid:

$$\widehat{U}_{\sigma_2} \cdot \widehat{U}_{\sigma_1} = \widehat{U}_{\sigma_2 \circ \sigma_1}, \qquad \text{or} \qquad U_{\sigma_2} * U_{\sigma_1} = U_{\sigma_2 \circ \sigma_1}.$$

Here $*$ is the noncommutative product in the function algebra over $\mathfrak{X}$ which corresponds to the operator representation (7.1):

$$\hat{f}\hat{g} = \widehat{f * g}$$

(see details and references in [1]).

Formula (7.5) is a Schrödinger type representation of the path σ, but one can define the Heisenberg type representation as well:

$$\widehat{U}_\sigma^{-1} \cdot \hat{f} \cdot \widehat{U}_\sigma = \widehat{\hat{\sigma}^* f}.$$

The mapping $\hat{\sigma}^*$ acts in the noncommutative algebra of functions over $\mathfrak{X}$ by the formula

$$(7.6) \qquad \hat{\sigma}^* = \underset{\to}{\mathrm{Exp}}\left(\frac{i}{2\hbar}\int_\sigma \mathrm{ad}_{\mathcal{H}^\hbar}\right), \qquad \mathrm{ad}_{\mathcal{H}^\hbar}(f) \overset{\mathrm{def}}{=} \mathcal{H}^\hbar * f - f * \mathcal{H}^\hbar,$$

and possesses the $\hbar$-expansion:

$$(7.7) \qquad\qquad\qquad \hat{\sigma}^* = [\sigma]^*\big(I + O(\hbar^2)\big),$$

where $[\sigma]$ is the classical path-symplectomorphism (4.1).

The definition (7.5) and formulas (7.6), (7.7) allow one to determine the *quantum holonomy* groups $\widehat{\mathcal{J}}_x$ (groups of unitary operators) and to calculate their Lie algebras. The quantum version of Theorem 5.1 is the following one.

Theorem 7.1. *Let $\mathfrak{X}$ be simply connected. For any point $x \in \mathfrak{X}$ the Lie algebra of the quantum holonomy group $\widehat{\mathcal{J}}_x$ is generated by the quantum curvature operators*

$$\widehat{\mathcal{R}}^\hbar_{xjk} = \frac{i}{4\hbar}\big[\widehat{\mathcal{H}}^\hbar_{xk}, \widehat{\mathcal{H}}^\hbar_{xj}\big],$$

where $\widehat{\mathcal{H}}^\hbar_{xk}$ are components of the quantum intrinsic Hamiltonian.

The *quantum curvature form* is naturally defined as

$$\widehat{\mathcal{R}}^\hbar = \frac{i}{8\hbar}[\widehat{\mathcal{H}}^\hbar \wedge_, \widehat{\mathcal{H}}^\hbar].$$

The quantum analog of (5.1) is the following:

$$(7.8) \qquad\qquad (\widehat{U}_{\partial\Sigma})^{-1} = \exp\left(-\frac{i}{\hbar}\int_\Sigma \widehat{\mathcal{R}}^\hbar + o(\Sigma)\right).$$

Following [1], we can derive the simplest semiclassical asymptotics of quantum paths, assuming, for simplicity, that the form ω is exact.

Let $\sigma \subset \mathfrak{X}$ be a path, and let for any $x \in \mathfrak{X}$ the symplectic mapping $s_x \circ [\sigma]$ have a unique fixed point $\tilde{x}$ (smoothly depending on x). Consider the membrane $\Sigma_\sigma(x)$ whose boundary consists of four pieces: a path c from $\tilde{x}$ to the beginning point of σ, the path σ, the path $[\sigma](c)$, and the Ether geodesics connecting $[\sigma](\tilde{x})$ with $\tilde{x}$ through the midpoint x.

Theorem 7.2. (i) *Under the above assumptions, the symbol U_σ of the quantum path (7.5) has the asymptotics*

$$(7.9) \qquad\qquad U_\sigma(x) = \exp\left\{\frac{i}{\hbar}\int_{\Sigma_\sigma(x)}\omega\right\}\varphi_\sigma(x) + O(\hbar),$$

where

$$(7.10) \qquad\qquad \varphi_\sigma(x) = 2^n \cdot \det\left(I - D(s_x \circ [\sigma])(\tilde{x})\right)^{-1/2}.$$

(ii) *The membrane $\Sigma_\sigma(x)$ in (7.9) can be chosen as a union of two pieces. The first piece Σ_σ is independent of x; its boundary is formed by the path σ and the Ether geodesic connecting the end-points of σ. Let us denote by 0_σ the mid-point of this Ether geodesic. The second piece is bounded by two paths c and $[\sigma](c)$, connecting the points $\tilde{x}$ and $[\sigma](\tilde{x})$ with the end-points of σ, and by two Ether geodesics through the mid-points x and 0_σ. The WKB-phase in (7.9) is represented as the sum*

$$(7.11) \qquad\qquad \int_{\Sigma_\sigma(x)}\omega = \int_{\Sigma_\sigma}\omega + \Phi^{[\sigma]}_{0_\sigma}(x),$$

where $\Phi^{[\sigma]}$ is the normalized generating function of $[\sigma]$ in the sense of [2] (see formula (3.3)).

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Remark 7.3. For any (closed or nonclosed) path σ the phase

$$(7.12) \qquad \Phi^{[\sigma]}(x) = \int_{\Sigma_\sigma(x)} \omega$$

appearing in (7.9) is a generating function of the symplectic mapping $[\sigma]$ in the sense of [2]. In particular, one has the relation

$$(7.13) \qquad d\Phi^{[\sigma]}(x) = \mathcal{H}_x([\sigma](\tilde{x})) = -\mathcal{H}_x(\tilde{x}),$$

where $\tilde{x}$, as above, denotes the fixed point of the mapping $s_x \circ [\sigma]$. One can rewrite relations (7.13) in the following form:

$$[\sigma](\tilde{x}) = l(x, d\Phi^{[\sigma]}(x)), \qquad \tilde{x} = r(x, d\Phi^{[\sigma]}(x)).$$

Here l and r are left and right mappings in the symplectic groupoid over $\mathfrak{X}$ (i.e., in a neighborhood of the zero section in $T^*\mathfrak{X}$), see [1, 2, 3].

Also note that in view of (7.5) the function U_σ satisfies the Cauchy problem for a Schrödinger-type evolution equation, and so, the phase $\Phi^{[\sigma]}$ (7.9), (7.12) satisfies the Hamilton–Jacobi equation

$$(7.14) \qquad \frac{\partial}{\partial \sigma''}\Phi^{[\sigma]}(x) + \frac{1}{2}\mathcal{H}_{\sigma''}\big(l(x, d\Phi^{[\sigma]}(x))\big) = 0.$$

Here $\frac{\partial}{\partial \sigma''}$ means the differential with respect to variations of the final endpoint of the path $\sigma = \{\sigma' \to \sigma''\}$.

8 Quantum evolution via translocation

Now let us answer the next question: what is the quantum version of translocations?

Let us consider an external Hamiltonian H on $\mathfrak{X}$ and the corresponding quantum operator $\widehat{H}$. The *quantum translocation* is the unitary operator

$$(8.1) \qquad \widehat{U}_y^t = \exp\left\{-\frac{i}{\hbar}H(y)t\right\}\widehat{U}_{X^t(y)},$$

where $\widehat{U}_{X^t(y)}$ is the quantum trajectory (quantum path) corresponding to the segment of the classical Hamiltonian trajectory (6.2).

Theorem 8.1. *The following permutation formula holds*:

$$(8.2) \qquad \left(-i\hbar\frac{\partial}{\partial t} + \widehat{H} \right) \cdot \widehat{U}_y^t = \widehat{U}_y^t \cdot \left(-i\hbar\frac{\partial}{\partial t} + \widehat{\mathbb{H}}_y^t \right),$$

where

$$\widehat{\mathbb{H}}_y^t \overset{\text{def}}{=} \widehat{X^t(y)}^* \left(H - \frac{1}{2}\dot{X}^t(y)\mathcal{H}_{X^t(y)}^\hbar \right) - H(y).$$

The quantum translocated Hamiltonian $\mathbb{H}_y^t$ differs from the classical translocated Hamiltonian H_y^t (6.3) by $O(\hbar^2)$:

$$(8.3) \qquad \mathbb{H}_y^t = H_y^t + O(\hbar^2) \qquad as \quad \hbar \to 0.$$

As a corollary, we obtain the following representation of the quantum Schrödinger type flow:

$$(8.4) \qquad \exp\left\{ -\frac{it}{\hbar}\widehat{H} \right\} = \widehat{U}_y^t \cdot \underset{\leftarrow}{\mathrm{Exp}}\left\{ -\frac{i}{\hbar}\int_0^t \widehat{\mathbb{H}}_y^t\, dt \right\}.$$

From Theorems 6.1 and 7.2 and formulas (8.1), (8.3), one can now easily derive the global semiclassical evolution of any quantum state localized at the point y.

Indeed, the operator $\widehat{\mathbb{H}}_y^t = \widehat{H}_y^t + O(\hbar^2)$ in (8.4), near the point y, looks like an oscillator Hamiltonian perturbed by cubic and higher-order terms. Thus if the Wigner function of the initial state is localized near y (for example, if it is a Gaussian type function), then this localized behavior will not change after application of the evolution operator $\mathrm{Exp}\{-\frac{i}{\hbar}\int_0^t \widehat{\mathbb{H}}_y^t\, dt\}$. For example, in the Gaussian case, as it follows from the general analysis [4, 6, 7, 8, 9], the quadratic part of the initial Gaussian exponent will remain quadratic and just transformed by the first variation system related to the Hamiltonian H_y^t; in our case this is system (6.9) over $T_y\mathfrak{X}$.

At the last stage one can integrate over all points $y \in \mathfrak{X}$ and calculate the quantum evolution for generic Cauchy data (as in [10]).

This is just the general scheme. Now let us present more details. Let $\Pi_{y,\hbar}^0$ be a family of functions (coherent states) over $\mathfrak{X}$, resolving the unity:

$$\frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} \Pi_{y,\hbar}^0\, dm(y) = 1,$$

and such that the rescaled functions

$$(8.5) \qquad \pi_{y,\hbar}^0(u) \overset{\text{def}}{=} \Pi_{y,\hbar}^0(\mathrm{Exp}_y(\sqrt{\hbar}u)), \qquad u \in T_y\mathfrak{X},$$

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depend regularly on $\hbar \to +0$.

We define the family of *dynamically deformed coherent states* $\Pi_{y,\hbar}^t$ by applications of the oscillator type evolution operators:

$$(8.6) \qquad \widehat{\Pi}_{y,\hbar}^t \stackrel{\text{def}}{=} \operatorname*{Exp}_{\leftarrow} \left\{ -\frac{i}{\hbar} \int_0^t \widehat{\mathbb{H}}_y^t \, dt \right\} \widehat{\Pi}_{y,\hbar}^0.$$

Using formula (8.4) and integrating over $y \in \mathfrak{X}$, we obtain a representation of the quantum evolution in the following form.

Theorem 8.2. *Let the initial state (Wigner function) $\rho_\hbar$ be resolved by the family of coherent states $\Pi_{y,\hbar}^0$ as follows:*

$$\rho_\hbar(x) = \frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} \Pi_{y,\hbar}^0(x) \rho_\hbar^0(y) \, dm(y),$$

where $\rho_\hbar^0$ is a distribution over $\mathfrak{X}$. Then the Schrödinger type evolution of the quantum state $\hat\rho_\hbar$ is given by the formula

$$(8.7) \quad \exp\left\{ -\frac{it}{\hbar}\widehat{H} \right\} \hat\rho_\hbar = \frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{X}} \rho_\hbar^0(y) \exp\left\{ -\frac{it}{\hbar} H(y) \right\} \widehat{U}_{X^t(y)} \widehat{\Pi}_{y,\hbar}^t \, dm(y).$$

In this formula, the quantum Hamilton trajectory $\widehat{U}_{X^t(y)}$ is determined by (7.5):

$$(8.8) \qquad \widehat{U}_{X^t(y)} = \operatorname*{Exp}_{\leftarrow}\left\{ -\frac{i}{2\hbar} \int_{X^t(y)} \widehat{\mathcal{H}}^\hbar \right\},$$

where $\widehat{\mathcal{H}}^\hbar$ is the quantum Ether Hamiltonian over $\mathfrak{X}$, and the multiplicative integral in (8.8) is taken along the Hamilton trajectory (6.2).

The semiclassical asymptotics of the quantum trajectory (8.8) is determined by Theorem 7.2:

$$(8.9) \qquad U_{X^t(y)}(x) = \exp\left\{ \frac{i}{\hbar} \int_{\Sigma_y^t(x)} \omega \right\} \varphi_y^t(x) + O(\hbar),$$

where

$$(8.10) \qquad \varphi_y^t(x) = 2^n \det\left(I - D(s_x \circ [X_y^t])(\tilde{x}_y^t) \right)^{-1/2}.$$

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In (8.10) *we denote by* $\tilde{x}_y^t$ *the fixed point of the symplectic transformation* $s_x \circ [X^t(y)]$, *where* s_x *are the reflection mappings related via* (2.3), (2.4) *to the Ether Hamiltonian* $\mathcal{H}$, *and* $[X^t(y)]$ *is the symplectic trajectory* (4.1):

$$
(8.11) \qquad\qquad [X_y^t]^* = \underset{\rightarrow}{\mathrm{Exp}}\left(\frac{1}{2} \int_{X^t(y)} \mathrm{ad}(\mathcal{H}) \right).
$$

The boundary of the membrane $\Sigma_y^t(x)$ *in* (8.9) *consists of an arbitrary path* c *from* $\tilde{x}_y^t$ *to* y, *the trajectory* (6.2) *from* y *to* $X^t(y)$, *the path* $[X^t(y)](c)$ (*with the opposite orientation*), *and the Ether geodesic connecting* $s_x(\tilde{x}_y^t)$ *with* $\tilde{x}_y^t$ *through the mid-point* x.

Formulas (8.6)–(8.10) generalize to the case of symplectic manifolds the well-known Gaussian type approximation program developed for Euclidean spaces (see, for instance, in [4, 7, 8, 9, 10, 11, 12]). We stress that in (8.7) the quantum translocation operator $\widehat{U}_{X^t(y)}$ is separated from the deformed coherent states $\widehat{\Pi}_{y,\hbar}^t$. Such a separation (or factorization) is a version of the so-called "interaction representation" in quantum physics. In our case the role of the leading free motion is played by the intrinsic Ether stream.

The Gaussian type approximation is a particular case of the general scheme described above.

Namely, let $\{\lambda_y^0\}$ be a positive Lagrangian distribution in the complexified tangent bundle $^{\mathbb{C}}T\mathfrak{X}$ (say, given by an almost complex structure on $\mathfrak{X}$). Denote by $\{\lambda_y^t\}$ a distribution obtained from $\{\lambda_y^0\}$ by rotating each Lagrangian plane λ_y^0 by means of the linearized Hamiltonian system (6.9). Let the functions $\pi_{y,\hbar}^0$ at $\hbar = 0$ be just Gaussian exponents assigned to λ_y^0 as in [7, 8]. Then the dynamically deformed coherent states $\Pi_{y,\hbar}^t$ (8.6) can also be presented in the form (8.5):

$$
(8.12) \qquad\qquad \Pi_{y,\hbar}^t(\mathrm{Exp}_y(\sqrt{\hbar}u)) \overset{\mathrm{def}}{=} \pi_{y,\hbar}^t(u),
$$

where $\pi_{y,\hbar}^t$ are regular in $\hbar \rightarrow +0$ and $\pi_{y,0}^t$ are just Gaussian exponents assigned to λ_y^t.

The function $\pi_{y,0}^t$ is evaluated explicitly by solving the Cauchy problem for the harmonic oscillator type equation over the tangent space $T_y\mathfrak{X}$ as follows:

$$
(8.13) \qquad \pi_{y,0}^t(u) = \underset{\leftarrow}{\mathrm{Exp}}\left\{ -i \int_0^t Q_y^\tau(u - i\Psi(y)\partial_u)\, d\tau \right\} \pi_{y,0}^0(u).
$$

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Here the oscillator type Hamiltonian is given by

$$Q_y^t(u) \overset{\text{def}}{=} \frac{1}{2}\langle V_y^{t*}\nabla^2 H(X^t(y))V_y^t u, u\rangle, \qquad u \in T_y\mathfrak{X}.$$

For example, if the connection Γ on $\mathfrak{X}$ is chosen so that the function H is covariantly quadratic (see (6.11)), then (8.13) reads

$$(8.14) \quad \pi_{y,0}^t(u) = \exp\left\{-\frac{it}{2}\langle\nabla^2 H(y)(u - i\Psi(y)\partial_u), u - i\Psi(y)\partial_u\rangle\right\}\pi_{y,0}^0(u).$$

Since $\pi_{y,0}^0$ is a Gaussian exponent, the application of the oscillator-type evolution operators in (8.13), (8.14) is made easily and explicitly by the well-known formulas.

Appendix.
Extension of Cartan's moving frame method

In this Appendix we represent a nonsymplectic version of the dynamic geometry from Sects. 2–6 making this theory to be applicable to arbitrary manifolds.

Let $\mathcal{M}$ be a manifold with an affine connection Γ. We denote by ∇ the covariant derivative with respect to Γ, i.e.,

$$(\nabla_j u)^k = \partial_j u^k + \Gamma_{sj}^k u^s$$

for any vector field u on $\mathcal{M}$.

Consider a differential one-form A on $\mathcal{M}$ with values in the space of vector fields on $\mathcal{M}$. This form can also be presented as a family of linear mappings $A_x(z) : T_x\mathcal{M} \to T_z\mathcal{M}$ (where $x, z \in \mathcal{M}$) so that the form A at the point x is given by

$$A_x = A_x(\cdot)_j\, dx^j, \qquad A_x(z)_j \in T_z\mathcal{M}.$$

We denote by $a = A|_{\text{diag}}$ the diagonal family of mappings $T_x\mathcal{M} \to T_x\mathcal{M}$ (where x is running over $\mathcal{M}$).

Assume that

$$(A.1) \qquad\qquad \partial A + \frac{1}{2}[A \underset{,}{\wedge} A] = 0,$$

where ∂ is the differential (acting in the space of forms on $\mathcal{M}$) and $[\cdot, \cdot]$ is the commutator (acting in the space of vector fields on $\mathcal{M}$). This is an analog of Eq. (2.1).

The field A generates a connection $\partial + A$ in the trivial bundle over $\mathcal{M}$ with the fiber $C^\infty(\mathcal{M})$. Condition (A.1) means that this connection is curvature free. Therefore, we obtain a big family of *internal translations* $g_{x,y} : \mathcal{M} \to \mathcal{M}$ labeled by pairs of points $x, y, \in \mathcal{M}$. By definition,

$$g^*_{x,y} = \mathop{\mathrm{Exp}}_{\to} \left(\int_y^x A \right).$$

The diffeomorphisms $g_{x,y}$ are analogs of Ether transformations (see Sect. 2). Their definition does not depend on a path connecting the points y and x because of the zero curvature condition (A.1).

Note that solutions A of Eq. (A.1) are essentially different from solutions of a zero curvature equation used in [14] following the deformation quantization approach [15]. The geometric objects used in [14, 15] are fiberwise (vertical) fields on $T\mathcal{M}$. In particular, the parallel translation from y to x generated by the zero curvature connection in the sense of [14] is a mapping $T_y\mathcal{M} \to T_x\mathcal{M}$ but not a diffeomorphism of $\mathcal{M}$ as in our approach. This is exactly the difference between kinematics and dynamics as it was explained in the Introduction.

Besides the zero curvature equation (A.1), we fix the boundary condition

(A.2)
$$\nabla' A \Big|_{\mathrm{diag}} = 0,$$

where ∇' denotes the covariant derivative of a vector-valued form with respect to the adjoint connection Γ', i.e., $(\nabla'_k A)^s_j \stackrel{\mathrm{def}}{=} D_k A^s_j + \Gamma^s_{kl} A^l_j$.

Combining (A.1) and (A.2), we obtain the following equation on the diagonal:

(A.3)
$$\delta a + \frac{1}{2}(a \underset{,}{\wedge} a) = 0.$$

Here δ denotes the adjoint covariant differential $\delta = d + \Gamma' \wedge$. The brackets $(\cdot, \cdot)$ are generated by the *torsion tensor* T of the connection Γ:

$$(u, v)^k \stackrel{\mathrm{def}}{=} u^s T^k_{sl} v^l.$$

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Equation (A.3) coincides with the first structural equation in the moving frame method due to E. Cartan. The solvability of (A.3) is guaranteed by the 1-st Bianchi's identity. This structural equation combines the covariant strength δa of the diagonal vector field a with the torsion of the connection Γ (see [13] about Cartan's ideas concerning the torsion).

Thus we see that the zero curvature equation (A.1) is a natural extension of the structural equation from $\mathcal{M}$ to $\mathcal{M} \times \mathcal{M}$, and the field A is an extension of the *Cartan field a*.

By choosing a field a, we set in addition to (A.2), the boundary condition

$$(\text{A.4}) \qquad\qquad A\Big|_{\text{diag}} = a.$$

Any solution of (A.1) satisfying the boundary conditions (A.2) and (A.4) will be called an *internal vector* field on $\mathcal{M}$.

Let us consider "trajectories" of the internal vector field A, that is, solutions of the equation

$$(\text{A.5}) \qquad\qquad \partial s = A(s).$$

Here $\partial = \partial/\partial x$ plays the role of time derivative, and the "trajectory" $s = s_x(z)$ of (A.5) is uniquely determined by the initial data

$$s_x(z)\Big|_{x=z} = z.$$

This is the analog of the Hamiltonian dynamics (2.4). Condition (A.1) guarantees the solvability of (A.5).

Assuming that the field A is complete, we obtain a family $\{s_x \mid x \in \mathcal{M}\}$ of diffeomorphisms of the manifold $\mathcal{M}$. For each $x \in \mathcal{M}$ the mapping s_x has the fixed point $z = x$, i.e.,

$$(\text{A.6}) \qquad\qquad s_x(x) = x.$$

Obviously, in the domain where $\det a(x) \neq 0$, this fixed point is isolated.

We shall say that the manifold $\mathcal{M}$ is endowed with an *inversive structure* if there is a family of diffeomorphisms $\{s_x \mid x \in \mathcal{M}\}$ possessing isolated fixed points (A.6). In general, the inversions s_x are not involutions ($s_x^2 \neq \text{id}$) and so the inversive structure is not a reflective structure.

Proposition A.1. (i) *Any inversive structure generates an internal vector field by the formula*

$$A_x(z) = (\partial_x s_x)(s_x^{-1}(z)).$$

The corresponding affine connection and the Cartan field are given by
(A.7)
$$\Gamma^l_{jk}(x) = -\frac{\partial^2 s_x^l(z)}{\partial z^m \partial x^r} \left[\frac{\partial s_x(z)}{\partial z}\right]^{-1\,m}_{\;k} \left[\frac{\partial s_x(z)}{\partial x}\right]^{-1\,r}_{\;j} \Bigg|_{z=x}, \qquad a(x)^l_j = \frac{\partial s_x^l(z)}{\partial x^j}\Bigg|_{z=x}.$$

The Cartan field a has no eigenvalues 0 or 1 in the spectrum.

(ii) *Let $A = A^+$ be an interval vector field on $\mathcal{M}$. Let s^+ be the family of inversions generated by A^+ via* (A.5). *Then there exists another solution of the zero curvature equation* (A.1):

(A.8)
$$A^- \overset{\text{def}}{=} -(Ds^+)^{-1} \cdot A^+(s^+),$$

or in more detail,

$$A^-_x(z) = -\left[\frac{\partial s_x^+(z)}{\partial z}\right]^{-1} \cdot A^+_x(s_x^+(z)).$$

The inversions s^- generated by A^- via (A.5) *are just $s^-_x = (s^+_x)^{-1}$. The boundary conditions are*

$$A^-\Big|_{\text{diag}} = a^-, \qquad (\nabla^-)'A^-\Big|_{\text{diag}} = 0.$$

Here $a^- = a^+/(a^+-I)$, and the covariant derivative ∇^- is taken with respect to the connection Γ^- defined by (A.7) *via the inversions s^-.*

The diagonal field a^- is a Cartan field, that is, the solution of (A.3) *corresponding to the connection Γ^-.*

(iii) *The internal translations $g_{x,y}$ are related to inversions as follows :*

$$g_{x,y} = s^+_x \circ s^-_y.$$

(iv) *Let $E^\pm = \text{Exp}^\pm_x(vt)$ denote the solutions of the equations*

$$\frac{d}{dt}E^\pm = \frac{1}{2}\langle A^\pm_x(E^\pm), v\rangle, \qquad E^\pm\Big|_{t=0} = x,$$

where $v \in T_x\mathfrak{X}$. Then

(A.9)
$$s^+_x(\text{Exp}^-_x(v)) = \text{Exp}^+_x(-v).$$

The curve $\sigma^+ \cup \sigma^- \subset \mathcal{M}$ composed of two pieces $\sigma^+ = \{\mathrm{Exp}_x^+(vt) \mid t \geq 0\}$ and $\sigma^- = \{\mathrm{Exp}_x^-(vt) \mid t \leq 0\}$ we call an *internal geodesic through the center x.* It follows from (A.9) that such internal geodesics are *inversive curves* with respect to the inversion $s^\pm$.

The pair of vector fields $A^\pm$ described in Proposition A.1, (ii) we call an *internal pair.*

In the case of a symplectic manifold $\mathcal{M}$ (with a symplectic form ω), the internal vector fields $A^\pm$ are Hamiltonian:

(A.10) $A_x^\pm(z) = D\mathcal{H}_x^\pm(z)\Psi(z),$ where $\Psi = \omega^{-1}$ is a Poisson tensor.

The inversions $s_s^\pm$ defined by the Hamiltonian systems (A.5) are symplecto-morphisms of $\mathcal{M}$.

Proposition A.2. *In the symplectic case the connection $\Gamma = \Gamma^\pm$ defined by (A.7) via inversions $s = s^\pm$ is a symplectic connection:*

$$\nabla^\pm \omega = 0.$$

The torsion $T = T^\pm$ of this connection obeys the cyclicity condition

(A.11) $\underset{j,k,l}{\mathfrak{S}}\, \omega_{js} T_{kl}^s = 0$ *(summation over cyclic permutations).*

The internal Hamiltonian $\mathcal{H} = \mathcal{H}^\pm$ (A.10) is reconstructed from inversions $s = s^\pm$ by the formula

(A.12) $\mathcal{H}_x(z) = \int_x^z \langle \partial s_x(s_x^{-1}(z)), \omega(z)\, dz \rangle.$

Here we assume that the zero boundary conditions hold on the diagonal:

(A.13) $\mathcal{H}_x^\pm(z)\Big|_{x=z} = 0.$

Two internal Hamiltonians $\mathcal{H}^+$ and $\mathcal{H}^-$ are related to each other by the inversion mapping:

(A.14) $\mathcal{H}_x^+(s_x^+(z)) = -\mathcal{H}_x^-(z).$

The derivatives of $\mathcal{H} = \mathcal{H}^\pm$ satisfy the following boundary condition on the diagonal:

(A.15) $\nabla_l \nabla_m \mathcal{H}_k \Big|_{\mathrm{diag}} = \omega_{ms} T_{lr}^s \Psi^{rj} \nabla_j \mathcal{H}_k \Big|_{\mathrm{diag}}.$

Here the covariant derivatives $\nabla = \nabla^{\pm}$ and the torsion tensor $T = T^{\pm}$ are assigned to the connection $\Gamma = \Gamma^{\pm}$.

So we see that, in the symplectic case, formula (A.12) replaces formula (2.7), relation (A.14) replaces the skew-symmetry relation (2.3), the boundary conditions (A.13), (A.14) replace conditions (2.2).

Note that the general situation where we deal with torsion and the inversive structure is, in fact, very common and presented in many examples. Nevertheless, let us now consider the torsion free case and the reflective structures on a manifold $\mathcal{M}$.

Proposition A.3. (i) *The inversions s_x defined by (A.5) are involutions iff the following skew-symmetry condition holds:*

(A.16)
$$A(s) = -Ds \cdot A,$$

or in detailed notation:

$$A_x(s_x(z)) = -\frac{\partial s_x(z)}{\partial z} A_x(z).$$

In this case, $A = A^+ = A^-$, the boundary condition holds:

(A.17)
$$a = A\Big|_{\text{diag}} = 2,$$

the connection Γ is given by (2.6), and Γ is torsion free: $T = 0$.

(ii) *For any torsion free affine connection on $\mathcal{M}$, the zero curvature equation (A.1) with boundary conditions (A.2), (A.17) and with the skew-symmetry condition (A.16) has a solution $A = A_x(z)$ at least near the diagonal $\text{diag} = \{x = z\}$ in $\mathcal{M} \times \mathcal{M}$, and in this case the (semiglobal) reflective structure is given on $\mathcal{M}$.*

An internal vector field A on $\mathcal{M}$ satisfying conditions (A.16), (A.17) will be called a *fundamental vector field*.

Using a fundamental vector field on $\mathcal{M}$ and multiplying it by the factor $\frac{1}{2}$, we obtain analogs of the results in Sects. 3–6. Indeed, it is clear that Hamiltonian systems (2.5), (3.1), (3.2) are easily extended to the case of general fundamental vector field. In particular, the notion of path-diffeomorphism is well defined and the analog of formula (4.1) holds:

$$[\sigma]^* = \mathop{\text{Exp}}_{\rightarrow}\left(\frac{1}{2} \int_{\sigma} A \right).$$

This is the parallel translation along the path σ with respect to the connection

$$\boldsymbol{\nabla}^0 = \partial + \frac{1}{2}A$$

acting in the trivial bundle over $\mathcal{M}$ with the fiber $C^\infty(\mathcal{M})$.

Theorems 2.2, 3.1, 4.2 hold in this general case as well. Instead of the Ether curvature (5.3), there appear the *dynamic curvature* vector fields

$$B_{jk} = \frac{1}{4}[A_{0k}, A_{0j}] = [\boldsymbol{\nabla}^0_j, \boldsymbol{\nabla}^0_k]$$

generating the Lie algebra of the dynamic holonomy group $\mathcal{L}_0$ of the manifold $\mathcal{M}$. The analogs of formulas in Theorem 5.1 are the following:

$$B^s_{jk}(0) = 0, \qquad (\nabla_m \ B_{jk})^s(0) = 2R^s_{mjk}(0).$$

Here the covariant derivative ∇ corresponding to the connection Γ.

The *dynamic curvature form* is defined as

$$B = \frac{1}{2}B_{jk}dx^k \wedge dx^j = \frac{1}{8}[A \underset{,}{\wedge} A].$$

This is the 2-form on $\mathcal{M}$ with values in vector fields on $\mathcal{M}$. The analog of formula (5.1) is

$$[\partial\Sigma]^{-1*} = \exp\left(\int_\Sigma B + o(\Sigma)\right).$$

In the presence of a general fundamental field the translocation operation is described as follows. We start from a dynamical system

(A.18)
$$\frac{d}{dt}X = u(X), \qquad X\Big|_{t=0} = x,$$

where u is a vector field on $\mathcal{M}$. The translation along trajectories of (A.18) is denoted by $X = X^t(x)$.

The segment of the trajectory (6.2) generates the path-diffeomorphism $[X^t(y)]\colon \mathcal{M} \to \mathcal{M}$. Here the point $y \in \mathcal{M}$ is arbitrarily fixed.

Then we define a time-dependent vector field on $\mathcal{M}$:

(A.19)
$$v^t_y \overset{\text{def}}{=} [X^t(y)]^{-1}_* \left(u - \frac{1}{2}\dot{X}^t(y)A_{X^t(y)}\right).$$

Here A is the fundamental field on $\mathcal{M}$, and the subscript $*$ means the standard operation on vector fields:

$$(\gamma_*^{-1}u)(z) \overset{\text{def}}{=} [d\gamma(z)]^{-1} u(\gamma(z)), \qquad \gamma : \mathcal{M} \to \mathcal{M}.$$

The field (A.19) determines the new dynamical system on $\mathcal{M}$:

$$(A.20) \qquad \frac{d}{dt}Z = v_y^t(Z), \qquad Z\Big|_{t=0} = x.$$

The solution of this system we denote by $Z = Z_y^t(x)$.

The transformation from (A.18) to (A.20) is the *translocation* to y. There is the following analog of Theorem 6.1.

Proposition A.4. (i) *Solutions of the original system* (A.18) *and the translocated system* (A.20) *are related to each other by the formula*

$$X^t = [X^t(y)] \circ Z_y^t.$$

Here $[X^t(y)]$ is the path-diffeomorphism corresponding to the trajectory of (A.18) *starting at y.*

(ii) *The point y is an equilibrium point of the translocated-to-y system* (A.20). *One has*

$$v_y^t(y) = 0, \qquad (\nabla v_y^t)(y) = M_y^t,$$

where

$$(A.21) \qquad M_y^t = (V_y^t)^{-1} \cdot \nabla u(X^t(y)) \cdot V_y^t.$$

Here the covariant derivative ∇ corresponds to the connection Γ.

(iii) *The differential of the solution to system* (A.18) *at the point y is given by formulas* (6.8), (6.9), *where the linear maps $M_y^t : T_y\mathcal{M} \to T_y\mathcal{M}$ are defined by* (A.21).

Corollary A.5. *If the vector field u and the connection Γ satisfy the consistency condition*

$$(A.22) \qquad \nabla_u(\nabla u) = 0$$

along the trajectory $X^t(y)$ (A.18) *starting at y, then*

$$(A.23) \qquad dX^t(y) = V_y^t \cdot \exp(t\nabla u(y)).$$

For the periodic trajectory, formula (A.23) *means the factorization of the monodromy matrix to the geometric monodromy (the holonomy of Γ) and the dynamic monodromy (with generator $\nabla u(y)$).*

If condition (A.22) holds at any point, then we say that the vector field u is *auto-linear* with respect to the connection Γ, or that the connection Γ *covariantly linearizes* the field u.

In respect of the factorization formula (A.23), the following natural questions arise: How a connection which covariantly linearizes a given vector field can be found? How a connection which covariantly linearizes a given vector field along a given trajectory can be found?

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